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IRREGULARITIES OF TWO-COLOURINGS OF THE $N\times N$ SQUARE LATTICE

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The first theorem of this paper concerns a result which says, in a *quantitative* form, that a two-colouring of the points of the $N\times N$ square lattice cannot be well-distributed simultaneously relative to all *line segments*. The proof is an adaptation of an analytic method of K. F. Roth.

Second we prove a surprisingly sharp result of the same spirit on *tilted rectangles*. The deeper part of the proof constitutes an immediate application of the so-called "integral equation method" due to W. M. Schmidt.

1. Introduction

A well-known theorem of K. F. Roth [7] states that two-colouring, red and blue, say, the integers from 1 to N in any fashion there always exists an arithmetic progression such that the difference of the numbers of red and blue terms in this progression has absolute value $\gg N^{1/4}$.

We use the typographically more convenient notation $f(N)\ll g(N)$ to mean $f(N)=O\left(g(N)\right)$, i.e., $|f(N)/g(N)|$ is bounded as N tends to infinity.

The above question belongs to the following general pattern. Let X be a finite set and $\mathscr{H}$ be a family of subsets of X. Given a two-colouring $f\colon X\to\{+1, -1\}$ of X, we call $\left|\sum_{y\in A} f(y)\right|$ the "error" of f relative to $A\in\mathscr{H}$. Set

$$D(\mathscr{H},f) = \max_{A\in\mathscr{H}}\left|\sum_{y\in A} f(y)\right|,$$

and introduce the "discrepancy" of $\mathscr{H}$ by the relation

$$\mathrm{Dis}\,(\mathscr{H}) = \min_{f} D(\mathscr{H},f),$$

where the minimum is taken over all two-colourings of X. Now the basic problem is to estimate and, if possible, to determine $\mathrm{Dis}\,(\mathscr{H})$.

Denote by $|H|$ the cardinality of the set H. We first list some general upper bounds for $\mathrm{Dis}\,(\mathscr{H})$ (cf. [3] and [1]).

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Theorem A. *Let* $\mathrm{Deg}(\mathscr{H})=\max_{y\in X}|\{A\in\mathscr{H}: y\in A\}|$, *where X denotes the vertex set of the hypergraph $\mathscr{H}$. Let $|\mathscr{H}|=n$, $|X|=k$ and $\mathrm{Deg}(\mathscr{H})=d$. Then*

(i) $$\mathrm{Dis}(\mathscr{H}) \ll (n\cdot\log n)^{1/2};$$

(ii) $$\mathrm{Dis}(\mathscr{H}) \leqq 2d-2;$$

(iii) $$\mathrm{Dis}(\mathscr{H}) \ll (d\cdot\log n)^{1/2}\log k. \blacksquare$$

We remark that each of these upper bounds would essentially follow from the validity of the following

Conjecture. $\mathrm{Dis}(\mathscr{H})\ll(d\cdot\log d)^{1/2}$.

We note that using a slight generalization of relation (iii) in Theorem A one can easily prove the existence of a two-colouring of the integers in the interval $[1, N]$ such that any arithmetic progression has "error" $\ll N^{1/4}(\log N)^{5/2}$. This shows that Roth's result cited above is essentially best possible (see [1]). Concerning an application of relation (ii) in Theorem A, see [2].

Our objective is to prove results in the spirit of Roth's theorem.

Theorem 1.1. (i) *Colouring the points of the $N\times N$ square lattice red and blue in any fashion, there exists a line segment with "error" $\gg N^{1/4-\varepsilon}$, that is, the difference of the numbers of red and blue points in this line segment has absolute value $\gg N^{1/4-\varepsilon}$.*

(ii) *On the other hand, there is a two-colouring of the $N\times N$ square lattice such that any line segment has "error" $\ll N^{1/3}(\log N)^{5/2}$.*

The proof of (i) will be an adaptation of *Roth's method* [7]. Probably the exponent 1/4 of N in the lower bound is not best possible. It would be worth improving on it.

We omit the proof of (ii) since it proceeds along the same lines as that of Theorem 1.3 in [1].

Since any line segment can be covered by a suitably thin *tilted rectangle* so that it contain no further lattice points, we obtain the following corollary of Theorem 1.1: For any two-colouring of the integer coordinate points of the square $[1, N]^2$ there exists a tilted rectangle in $[1, N]^2$ with "error" $\gg N^{1/4-\varepsilon}$. However, for *arbitrary* tilted rectangles (not necessarily contained in $[1, N]^2$) *Schmidt's integral equation method* (cf. [8], [9], [10]) yields an essential improvement on this lower bound.

We mention that, using this ingenious (and mysterious) method, W. M. Schmidt [9] has proved the following basic result in the theory of irregularities of point-distributions: Given N points $p_1, \ldots, p_N$ in the unit square $[0, 1]^2$, there exists a tilted rectangle R (not necessarily contained in $[0, 1]^2$) such that

$$\Big|\sum_{i:\, p_i\in R} 1 - N\cdot(\text{the area of } R\cap[0, 1]^2)\Big| \gg N^{1/4-\varepsilon}.$$

Note that here the exponent 1/4 of N is best possible, see [2], Section 4.

Theorem 1.2. (i) *For any two-colouring of the $N\times N$ square lattice there exists a tilted rectangle with "error" $\gg N^{1/2}(\log N)^{-1/2}$.*

(ii) *In the opposite direction, there is a two-colouring of $N\times N$ such that any tilted rectangle has "error" $\ll N^{1/2}(\log N)^{1/2}$.*

Though the proof of (i) will be a direct application of Schmidt's integral equation method, for the sake of completeness we outline it. Note that Schmidt's method gives only the lower bound $N^{1/2-\varepsilon}$. To improve N^ε to $(\log N)^{1/2}$ we shall use a technical modification due to G. Herman [6].

The upper bound will be an application of the "probabilistic method" (cf. Erdős—Spencer [5]).

Observe that the case of rectangles with sides parallel to the coordinate axes (*aligned* rectangles) is trivial: the chess-board type two-colouring of the set $\mathbf{Z}^2$ of all integer coordinate points has "error" $\leqq 1$ for any aligned rectangle. Now the following natural question arises: Is it true that, given any N-element set in the plane, one can find a two-colouring of it such that every aligned rectangle has "error" $\leqq 1$? The answer is negative, see [2].

2. Proof of Theorem 1.1, (i)

We use α, β to denote real numbers and $e(\alpha)$ to denote $\exp\{2\pi i\alpha\}$. Following Roth [7] we shall employ the theory of complex variables. Let us be given a two-colouring $h(\mathbf{l})=\pm 1$ of the integer coordinate points $\mathbf{l}=(l_1, l_2)$ of the square $[1, N]^2$. Let

$$E(\alpha_1, \alpha_2) = \sum_{l_1=1}^{N}\sum_{l_2=1}^{N} h(l_1, l_2)e(l_1\alpha_1+l_2\alpha_2) \quad \text{and} \quad F(\beta) = \sum_{k=0}^{K-1} e(k\beta),$$

where the parameter K will be fixed later. As learned from Roth [7] we shall prove the theorem by comparing upper and lower bounds for the expression

$$G = \int_0^1\int_0^1 \sum_{\mathbf{q}=(q_1,q_2)}^{*} |E(\alpha_1, \alpha_2)\cdot F(q_1\alpha_1+q_2\alpha_2)|^2\, d\alpha_1\, d\alpha_2.$$

Here the summation Σ^* is extended over all pairs (q_1, q_2) such that $1\leqq |q_i|\leqq Q$, $i=1, 2$, and q_1 and q_2 *are prime to each other*. The parameter Q will be specified later.

To obtain a lower estimate for G, we need a technical lemma on exponential sums.

Lemma 2.1. *Given any* $\varepsilon>0$ *there is a threshold* $Q_0(\varepsilon)$ *such that for* $Q>Q_0(\varepsilon)$

$$\min_{0\leqq\alpha_1\leqq 1}\ \min_{0\leqq\alpha_2\leqq 1}\ \sum_{(q_1,q_2)}^{*} |F(q_1\alpha_1+q_2\alpha_2)|^2 \gg Q^{3(1-\varepsilon)},$$

where $F(\beta)=\sum_{k=0}^{K-1} e(k\beta)$ *with* $K=[Q^{1-\varepsilon}/6]$ *(integral part).*

Note that omitting the restriction "q_1 and q_2 are prime to each other" in the summation above, the lemma becomes a trivial consequence of Dirichlet's theorem in diophantine approximation (see Roth [7]). Lemma 2.1 will be verified in the next section.

Clearly

(1) $$G \geqq \int_0^1 \int_0^1 |E(\alpha_1, \alpha_2)|^2 \, d\alpha_1 \, d\alpha_2 \cdot \min_{0 \leqq \alpha_1 \leqq 1} \min_{0 \leqq \alpha_2 \leqq 1} \sum_{(q_1, q_2)}^{*} |F(q_1\alpha_1 + q_2\alpha_2)|^2,$$

and by Parseval's equation

(2) $$\int_0^1 \int_0^1 |E(\alpha_1, \alpha_2)|^2 \, d\alpha_1 \, d\alpha_2 = \sum_{l_1=1}^{N} \sum_{l_2=1}^{N} h^2(l_1, l_2) = N^2.$$

Set $K=[Q^{1-\varepsilon}/6]$ and apply Lemma 2.1. By (1) and (2)

(3) $$G \gg N^2 Q^{3(1-\varepsilon)}.$$

To estimate G from above, let $P(\mathbf{q}, \mathbf{l})$ denote the set of integer coordinate points $\mathbf{l}' \in [1, N]^2$ which are representable as $\mathbf{l}' = \mathbf{l} - k \cdot \mathbf{q}$, where the coordinates q_1, q_2 of q are prime to each other and $0 \leqq k \leqq K-1$. Since q_1 and q_2 are coprime, the elements of $P(\mathbf{q}, \mathbf{l})$ are on some line segment $L(\mathbf{q}, \mathbf{l})$ containing exactly these integer coordinate points, i.e.,

$$L(\mathbf{q}, \mathbf{l}) \cap P(\mathbf{q}, \mathbf{l}) = L(\mathbf{q}, \mathbf{l}) \cap \mathbf{Z}^2.$$

Set

$$D(\mathbf{q}, \mathbf{l}) = D(\mathbf{q}, \mathbf{l};\ h) = \sum_{\mathbf{l}' \in P(\mathbf{q}, \mathbf{l})} h(\mathbf{l}').$$

Clearly $|D(\mathbf{q}, \mathbf{l})|$ is the "error" of the line segment $L(\mathbf{q}, \mathbf{l})$. The reader can easily verify the following basic identity:

$$E(\alpha_1, \alpha_2) \cdot F(q_1\alpha_1 + q_2\alpha_2) = \sum_{\mathbf{l}=(l_1, l_2)} D(\mathbf{q}, \mathbf{l}) \cdot e(l_1\alpha_1 + l_2\alpha_2).$$

Thus, by Parseval's equality

$$\int_0^1 \int_0^1 |E(\alpha_1, \alpha_2) \cdot F(q_1\alpha_1 + q_2\alpha_2)|^2 \, d\alpha_1 \, d\alpha_2 = \sum_{\mathbf{l}} D^2(\mathbf{q}, \mathbf{l}).$$

Summing over $\mathbf{q}$ we obtain

(4) $$G = \sum_{\mathbf{q}}^{*} \sum_{\mathbf{l}} D^2(\mathbf{q}, \mathbf{l}).$$

From the definition of $D(\mathbf{q}, \mathbf{l})$ it follows that $D(\mathbf{q}, \mathbf{l}) = 0$ if we have either $l_1 \notin (-K \cdot Q, N + K \cdot Q)$ or $l_2 \notin (-K \cdot Q, N + K \cdot Q)$. Thus by (4) we conclude that

$$G \ll Q^2 \cdot M^2 \cdot \Delta^2,$$

where $M = N + K \cdot Q$ and $\Delta = \max_{\mathbf{q}, \mathbf{l}} |D(\mathbf{q}, \mathbf{l})|$. This means that there exists a line segment with "error"

(5) $$\Delta \gg G^{1/2}/(Q \cdot M).$$

Let $Q = [N^{1/2}]$. Then $M \ll N$; thus by (3) and (5) we have

$$\Delta \gg Q^{1/2 - 2\varepsilon} = [N^{1/2}]^{1/2 - 2\varepsilon} \gg N^{1/4 - \varepsilon},$$

which completes the proof of Theorem 1.1, (i). ∎

3. Proof of Lemma 2.1.

The proof is due to I. Z. Ruzsa. We need the following simple fact

$$(6)\qquad \left|\sum_{k=0}^{K-1} e(k\beta)\right| = K\cdot\left|\frac{\sin\pi\cdot K\cdot\beta}{K\cdot\|\beta\|}\right|\cdot\left|\frac{\|\beta\|}{\sin\pi\beta}\right| \geqq \frac{2}{\pi}K \text{ for all } \beta \text{ with } \|\beta\| \leqq \frac{1}{2K}.$$

Here and in what follows $\|\beta\|$ denotes the distance from β to the nearest integer. Denote by g.c.d.(a, b) the greatest common divisor of the integers a and b.

Without loss of generality we may assume that $\alpha_1 \leqq \alpha_2 \neq 0$. By Dirichlet's theorem in diophantine approximation there is a rational number r/q such that g.c.d.$(r, q)=1$, $1\leqq|r|\leqq|q|\leqq Q$ and

$$|\alpha_1/\alpha_2 + r/q| \leqq \frac{1}{q\cdot Q}.$$

Hence

$$|q\alpha_1 + r\alpha_2| \leqq \frac{\alpha_2}{Q} \leqq \frac{1}{Q},$$

that is, there exists at least *one* coprime pair (q, r) with the properties $1\leqq|q|, |r|\leqq Q$ and $\|q\alpha_1 + r\alpha_2\| \leqq 1/Q$.

Using the box principle one can easily prove the existence of at least $2Q$ different (not necessarily coprime) pairs (a_i, b_i) with $1\leqq|a_i|, |b_i|\leqq Q$ and $\|a_i\alpha_1+b_i\alpha_2\|\leqq \leqq 2/Q$. Omitting those pairs (a_i, b_i) for which $a_i/b_i=r/q$, we obtain at least Q pairs (a_j, b_j) with the additional property $a_j/b_j\neq r/q$. Now fix such a pair (a_j, b_j). Assume that Q is sufficiently large depending on ε. We claim that there is an integer k_j such that $1\leqq k_j\leqq Q^{\varepsilon}$ and g.c.d.$(q+k_ja_j, r+k_jb_j)=1$. The following is the very lemma we need.

Lemma 3.1. *Let us be given integers q, r, a, b with $1\leqq|q|, |r|, |a|, |b|\leqq Q$ and* g.c.d. $(q, r)=1$. *Then for $Q>Q_0(\varepsilon)$ there exists an integer k such that $1\leqq k\leqq Q^{\varepsilon}$ and* g.c.d. $(q+ka, r+kb)=1$.

Proof. For notational convenience let $H=qb-ra$, $d=$g.c.d.(a, q). Let $p_1', \ldots, p_s'$ denote those prime divisors of H/d which are relatively prime to a. Let $p_1'', \ldots, p_t''$ denote those prime divisors of d which are relatively prime to b. Let $P'\subset\{p_1', \ldots, p_s'\}$, $P''\subset\{p_1'', \ldots, p_t''\}$ be arbitrary subsets. Introduce the notation

$$S(P'\cup P'') = \Big\{k\colon 1\leqq k\leqq Q^{\varepsilon},\ q+ka \text{ is divisible by } \prod_{p'\in P'} p' \text{ and}$$
$$r+kb \text{ is divisible by } \prod_{p''\in P''} p''\Big\},$$

$$S^* = \Big\{k\colon 1\leqq k\leqq Q^{\varepsilon},\ q+ka \text{ is \textit{not} divisible by any } p_i',\ 1\leqq i\leqq s \text{ and}$$
$$r+kb \text{ is \textit{not} divisible by any } p_j'',\ 1\leqq j\leqq t\Big\}.$$

It is easy to see that

$$S^* \subseteqq \{k\colon 1\leqq k\leqq Q^{\varepsilon},\ \text{g.c.d.}(q+ka, r+kb) = 1\}.$$

Indeed, let $k\in S^*$ and assume that, on the contrary, p is a common prime divisor of $q+ka$ and $r+kb$. Since $H=b(q+ka)-a(r+kb)$, H is divisible by p. First assume that d is not divisible by p. Since $d=$g.c.d.(a, q), it follows that a is not divisible by p. Thus $p\in\{p_1', \dots, p_s'\}$. Since $k\in S^*$, we conclude that $q+ka$ is *not* divisible by p, a contradiction. Now assume that p is a divisor of d. Since the case $p\in\{p_1'', \dots, p_t''\}$ is impossible by $k\in S^*$, we obtain that p is a divisor of b. Since p is a divisor of $r+kb$, p is a divisor of r as well. On the other hand, $d=$g.c.d. (a, q), hence q is divisible by p. But this is a contradiction, since r and q are prime to each other.

By the inclusion-exclusion principle (sieve of Eratosthenes)

$$(7)\qquad |S^*| = |S(\emptyset)|-\Big(\sum_{1\le i\le s}|S(\{p_i'\})|+\sum_{1\le j\le t}|S(\{p_j''\})|\pm\dots =$$

$$= \sum_{l=0}^{s+t}(-1)^l\sum_{i=0}^{l}\sum_{\substack{|P'|=i\\|P''|=l-i}}|S(P'\cup P'')|.$$

Observe that for each $P'\subset\{p_1', \dots, p_s'\}$, $P''\subset\{p_1'', \dots, p_t''\}$

$$\Big||S(P'\cup P'')|-Q^\varepsilon\cdot\prod_{p'\in P'}\frac{1}{p'}\cdot\prod_{p''\in P''}\frac{1}{p''}\Big|\le 1.$$

Thus by (7) we have

$$|S^*| = Q^\varepsilon\cdot\Big\{1-\Big(\sum_{i=1}^{s}\frac{1}{p_i'}+\sum_{j=1}^{t}\frac{1}{p_j''}\Big)\pm\dots\Big\}+O(2^{s+t}) =$$

$$= Q^\varepsilon\cdot\prod_{i=1}^{s}\Big(1-\frac{1}{p_i'}\Big)\cdot\prod_{j=1}^{t}\Big(1-\frac{1}{p_j''}\Big)+O(2^{s+t})\ge$$

$$\ge Q^\varepsilon\cdot 2^{-s-t}+O(2^{s+t})\ge 1.$$

In the last step we used the well-known fact that the number of prime divisors of H is $O(\log H/\log\log H)\le O(\log Q/\log\log Q)$ ∎.

By the application of Lemma 3.1 it follows that there are at least Q (not necessarily different) coprime pairs

$$(q_j, r_j) = (q+k_ja_j, r+k_jb_j)$$

such that $1\le|q_j|, |r_j|\le Q^{1+\varepsilon}$ and $\|q_j\alpha_1+r_j\alpha_2\|=\|q\alpha_1+r\alpha_2\|+k_j\|a_j\alpha_1+b_j\alpha_2\|\le \le Q^{-1}+2Q^{-1+\varepsilon}\le 3Q^{-1+\varepsilon}$. Since $1\le k_j\le Q^\varepsilon$, there are at least $Q^{1-\varepsilon}$ pairs (q_j, r_j) with the *same* value of k_j. Observe that these pairs are already different. Now the proof of Lemma 2.1 can be immediately completed by the application of (6). ∎

4. Proof of relation (i) in Theorem 1.2; the application of Schmidt's integral equation method

We shall apply Schmidt's notation. Denote by $R(\mathbf{u})$ the rectangle of points $\mathbf{x}=(x_1, x_2)$ with $0\le x_1\le u_1$ (or $u_1\le x_1\le 0$), $0\le x_2\le u_2$ (or $u_2\le x_2\le 0$) where $\mathbf{u}=(u_1, u_2)$. Denote by τ_φ the rotation by angle φ about the origin $\mathbf{O}$. Write $R(\mathbf{u}, \mathbf{v}, \varphi)$ for the rectangle of points $\tau\varphi\mathbf{x}+\mathbf{v}$ with $\mathbf{x}\in R(\mathbf{u})$. It is easy to see that all rectangles of diameter 1 are of the form

$$R(\mathbf{w}(\psi), \mathbf{v}, \varphi), \quad 0 \leqq \psi, \varphi \leqq 2\pi,$$

where $\mathbf{w}(\psi)=(\cos\psi, \sin\psi)$.

Let P^* denote the set of points $\left(\frac{i}{N}, \frac{j}{N}\right)$, where i and j run over all integers. Let Q be a finite subset of the unit sqare $U^2=[0,1)\times[0,1)$. Denote by Q^* the set of points $\mathbf{q}+\mathbf{l}$, $\mathbf{q}\in Q$, where $\mathbf{l}$ runs over all integer coordinate points. Thus Q^* and P^* are "periodic" sets. Given any bounded set A in the plane, write $Z(A; Q^*)$ and $Z(A; P^*)$ for the number of points of Q^* in A and the number of points of P^* in A, respectively. Set

$$D(A) = Z(A; P^*)/2 - Z(A; Q^*).$$

Let $K(r, \mathbf{c})$ be the disc with radius r and centre $\mathbf{c}$, i.e., the set of points $\mathbf{x}$ with $|\mathbf{x}-\mathbf{c}|<r$ ($|\mathbf{a}-\mathbf{b}|$ denotes the Euclidean distance of the points $\mathbf{a}$ and $\mathbf{b}$).

Let

$$D(r, \mathbf{c}) = D(K(r, \mathbf{c})) = Z(K(r, \mathbf{c}); P^*)/2 - Z(K(r, \mathbf{c}); Q^*)$$

and

$$E(r, s) = \iint_{U^2} D(r, \mathbf{c}) D(s, \mathbf{c})\, d\mathbf{c}.$$

In the first step we shall prove

Lemma 4.1. *Suppose that* $0\leqq\delta\leqq 1/2$. *Then*

$$\frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, D^2(R(\delta\cdot\mathbf{w}(\psi), \mathbf{v}, \varphi)) = \int_0^{1/2} E(\delta r, \delta r) f(r)\, dr,$$

where $f(r) \gg\ll r^{-1}$. *The notation* $f\gg\ll g$ *means that both* $f\gg g$ *and* $f\ll g$.

Second we shall verify

Lemma 4.2. *Suppose that* $0\leqq\delta\leqq 1/2$. *Given any* $0<\alpha<1$, *we have*

$$\int_0^{1/2} E(\delta r, \delta r)\, g(r)\, dr = \underset{r+s\leqq 1}{\int_0^1\int_0^1} E(\delta r, \delta s)|r-s|^{-\alpha}|r+s|^{-(1-\alpha)}\, dr\, ds,$$

where $g(r)\gg\ll r^{-1-\alpha}$. *Here all constants involved are bounded above and below by positive constants independent of* α.

Following Schmidt we now explain how Lemma 4.1 and 4.2 will be used to establish the desired lower bound in Theorem 1.2, (i). First observe that

$$|2E(r, s)| \leqq \iint_{U^2} 2|D(r, \mathbf{c})\cdot D(s, \mathbf{c})|\, d\mathbf{c} \leqq$$

$$\leqq \iint_{U^2} (D^2(r, \mathbf{c}) + D^2(s, \mathbf{c}))\, d\mathbf{c} = E(r, r) + E(s, s).$$

Therefore a simple calculation shows that

$$\Big| \int_0^1 \int_0^1_{r+s\le 1} E(r/4, s/4)\cdot |r-s|^{-\alpha} \cdot |r+s|^{-(1-\alpha)}\, dr\, ds \Big| \le$$

$$\le \int_0^1 \int_0^1_{r+s\le 1} E(r/4, r/4) |r-s|^{-\alpha} |r+s|^{-(1-\alpha)}\, dr\, ds =$$

$$= \int_0^1 dr\, E(r/4, r/4) \int_0^{\frac{1}{r}-1} (1-t)^{-\alpha}(1+t)^{-(1-\alpha)}\, dt \ll$$

$$\ll \int_0^1 \left(\frac{1}{1-\alpha} + \log\frac{1}{r}\right) E(r/4, r/4)\, dr = 4 \int_0^{1/4} \left(\frac{1}{1-\alpha} + \log\frac{1}{4r}\right) E(r, r)\, dr.$$

By Lemma 4.2,

$$\int_0^{1/4} \left(\frac{1}{1-\alpha} + \log\frac{1}{4r}\right) E(r, r)\, dr \gg \int_0^{1/4} r^{-1-\alpha} E(r, r)\, dr.$$

Choosing $\alpha = 1-(\log N)^{-1}$ and $\delta = 1/2$, by Lemma 4.1. we conclude that

$$\frac{1}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, D^2\big(R(\delta\cdot \mathbf{w}(\psi), \mathbf{v}, \varphi)\big) =$$

$$= \int_0^{1/2} E(\delta r, \delta r) f(r)\, dr \gg (\log N)^{-1} \int_0^{1/4} \left(\log N + \log\frac{1}{4r}\right) E(r, r)\, dr \gg$$

$$\gg (\log N)^{-1} \int_0^{1/4} r^{-1-\alpha} E(r, r)\, dr.$$

To estimate $E(r, r)$ from below, we shall use the obvious fact: for $\frac{1}{4N} < r \le \frac{1}{2N}$ and $\mathbf{c} \in \bigcup_{\mathbf{p}\in P^*} K\left(\frac{1}{4N}, \mathbf{p}\right) \cap U^2$ we have either $D(r, \mathbf{c}) = \frac{1}{2}$ or $D(r, \mathbf{c}) \le -\frac{1}{2}$. Therefore, the set $\{\mathbf{c}\in U^2 : |D(r, \mathbf{c})| \ge 1/2\}$ has area $\ge c_2 > 0$ if only $\frac{1}{4N} < r \le \frac{1}{2N}$. Hence $E(r, r) \gg 1$ for $1/4N < r \le 1/2N$, and

$$\int_0^{1/4} r^{-1-\alpha} E(r, r)\, dr \gg \int_{1/(4N)}^{1/(2N)} r^{-2}\, dr \gg N.$$

Summarizing, we obtain

$$\frac{1}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, D^2\big(R(\delta\cdot \mathbf{w}(\psi), \mathbf{v}, \varphi)\big) \gg \frac{N}{\log N},$$

where $\delta=1/2$. From this it immediately follows that, given *any* $Q\subset U^2$, there exists a rectangle R of diameter 1/2 with

$$|Z(R;\ P^*)/2-Z(R;\ Q^*)| \gg N^{1/2}\cdot(\log N)^{-1/2}.$$

Now let us be given any two-colouring $h(\mathbf{l})=\pm 1$ of the integer coordinate points $\mathbf{l}$ of $[0, N)\times[0, N)$. Set

$$Q = \left\{\left(\frac{i}{n}, \frac{j}{n}\right): h(i,j) = +1\right\}.$$

Applying the previous statement to this Q, we obtain Theorem 1.2, (i). ∎

It remains to verify Lemmas 4.1 and 4.2. First we outline a proof of Lemma 4.1, referring to [8] (the reader may also follow an argument using [10]). We shall employ Schmidt's notation. Set

$$f_R(\mathbf{u}, \mathbf{v}, \varphi; \mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x}\in R(\mathbf{u}, \mathbf{v}, \varphi), \\ 0 & \text{otherwise,} \end{cases}$$

that is, for fixed $\mathbf{u}$, $\mathbf{v}$ and $\varphi, f_R(\mathbf{u}, \mathbf{v}, \varphi)$ is the characteristic function of the rectangle $R(\mathbf{u}, \mathbf{v}, \varphi)$. Note that

$$f_R(\mathbf{u}, \mathbf{v}+\mathbf{c}, \varphi; \mathbf{x}+\mathbf{c}) = f_R(\mathbf{u}, \mathbf{v}, \varphi; \mathbf{x}).$$

Next set

$$g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{x}) = \sum_{\mathbf{l}} f_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{x}+\mathbf{l}),$$

where the summation is taken over all lattice points $\mathbf{l}$. Write

$$h_R(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{x})\, g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{y}). \tag{8}$$

Simple argument shows (see p. 350 of [8]) that for $\mathbf{u}$ fixed, $|\mathbf{u}|\leqq 1/2$, $h_R(\mathbf{u}, \mathbf{x}, \mathbf{y})$ as a function of $\mathbf{x}$ and $\mathbf{y}$ depends only on the "distance modulo 1" of $\mathbf{x}$ and $\mathbf{y}$. More exactly, let $\omega(\mathbf{x}, \mathbf{y})=\min_{\mathbf{l}} |\mathbf{x}-\mathbf{y}-\mathbf{l}|$, where the minimum is taken over all lattice points. There is a function $K_R(\mathbf{u}, \omega)$ such that

$$h_R(\mathbf{u}, \mathbf{x}, \mathbf{y}) = K_R\big(\mathbf{u}, \omega(\mathbf{x}, \mathbf{y})\big) \quad \text{for} \quad |\mathbf{u}| \leqq 1/2. \tag{9}$$

Set $P=P^*\cap U^2=\left\{\left(\frac{i}{N}, \frac{j}{N}\right):\ 0\leqq i,\ j\leqq N-1\right\}$. Since

$$Z\big(R(\mathbf{u}, \mathbf{v}, \varphi);\ P^*\big) = \sum_{\mathbf{p}\in P}\sum_{\mathbf{l}} f_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p}+\mathbf{l}) = \sum_{\mathbf{p}\in P} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p})$$

and

$$Z\big(R(\mathbf{u}, \mathbf{v}, \varphi);\ Q^*\big) = \sum_{\mathbf{q}\in Q} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{q}),$$

we have

$$(10)\qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v} D^2(R(\mathbf{u}, \mathbf{v}, \varphi)) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}(Z(R(\mathbf{u}, \mathbf{v}, \varphi);\ P^*)/2 - Z(R(\mathbf{u}, \mathbf{v}, \varphi);\ Q^*))^2 =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\Big(\frac{1}{2}\sum_{\mathbf{p}\in Q} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p}) - \sum_{\mathbf{q}\in P} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{q})\Big)^2 =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\Big(\frac{1}{4}\sum_{\mathbf{p}'\in P}\sum_{\mathbf{p}''\in P} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p}')g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p}'') +$$

$$+ \sum_{\mathbf{q}'\in Q}\sum_{\mathbf{q}''\in Q} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{q}')g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{q}'') -$$

$$- \sum_{\mathbf{p}\in P}\sum_{\mathbf{q}\in Q} g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{p})g_R(\mathbf{u}, \mathbf{v}, \varphi;\ \mathbf{q})\Big).$$

By (8), (9) and (10)

$$(11)\qquad \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, D^2(R(\mathbf{u}, \mathbf{v}, \varphi)) = \frac{1}{4}\sum_{\mathbf{p}'\in P}\sum_{\mathbf{p}''\in P} K_R(\mathbf{u}, \omega(\mathbf{p}', \mathbf{p}'')) +$$

$$+ \sum_{\mathbf{q}'\in Q}\sum_{\mathbf{q}''\in Q} K_R(\mathbf{u}, \omega(\mathbf{q}', \mathbf{q}'')) - \sum_{\mathbf{p}\in P}\sum_{\mathbf{q}\in Q} K_R(\mathbf{u}, \omega(\mathbf{p}, \mathbf{q})),$$

provided that $|\mathbf{u}| \leqq 1/2$.

We now consider rectangles of diameter $\delta \leqq 1/2$. Substituting $\mathbf{u} = \delta \cdot \mathbf{w}(\psi)$ in (11) and integrating both sides with respect to ψ from 0 to 2π, we obtain

$$(12)\qquad \frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\mathbf{v}\, D^2(R(\delta\mathbf{w}(\psi), \mathbf{v}, \varphi)) =$$

$$= \frac{1}{4}\sum_{\mathbf{p}'\in P}\sum_{\mathbf{p}''\in P} l_R(\delta, \omega(\mathbf{p}', \mathbf{p}'')) + \sum_{\mathbf{q}'\in Q}\sum_{\mathbf{q}''\in Q} l_R(\delta, \omega(\mathbf{q}', \mathbf{q}'')) -$$

$$- \sum_{\mathbf{p}\in P}\sum_{\mathbf{q}\in Q} l_R(\delta, \omega(\mathbf{p}, \mathbf{q})),$$

where

$$l_R(\delta, \omega) = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, K_R(\delta\mathbf{w}(\psi), \omega).$$

Set

$$f(r, \mathbf{c}; \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x}\in K(r, \mathbf{c}), \\ 0 & \text{otherwise,} \end{cases}$$

that is, or fixed r and $\mathbf{c}$, $f(r, \mathbf{c}; \mathbf{x})$ is the characteristic function of the disc $K(r, \mathbf{c})$. Next set

$$g(r, \mathbf{c};\ \mathbf{x}) = \sum_{\mathbf{l}} f(r, \mathbf{c};\ \mathbf{x}+\mathbf{l}),$$

where the summation is taken over all lattice points. There is a function $K(r, \omega)$ such that for $0 \leqq r \leqq 1/4$

$$\int_{U^2} g(r, \mathbf{c};\ \mathbf{x})\, g(r, \mathbf{c};\ \mathbf{y})\, d\mathbf{c} = K(r, \omega(\mathbf{x}, \mathbf{y})), \tag{13}$$

i.e., the left hand side of (13) as a function of $\mathbf{x}$ and $\mathbf{y}$ depends only on the "distance modulo 1" of $\mathbf{x}$ and $\mathbf{y}$. Actually, $K(r, \omega)$ is the area of the intersection of two discs with equal radius r whose centres have distance ω. Similarly as under (10) and (11), by (13) we have

$$E(r, r) = \frac{1}{4} \sum_{\mathbf{p}' \in P} \sum_{\mathbf{p}'' \in P} K(r, \omega(\mathbf{p}', \mathbf{p}''))+ \tag{14}$$

$$+ \sum_{\mathbf{q}' \in Q} \sum_{\mathbf{q}'' \in Q} K(r, \omega(\mathbf{q}', \mathbf{q}'')) - \sum_{\mathbf{p} \in P} \sum_{\mathbf{q} \in Q} K(r, \omega(\mathbf{p}, \mathbf{q})) \quad \text{for} \quad 0 \leqq r \leqq 1/4.$$

We recall a result of [8].

Lemma 4.3. *Suppose* $0 \leqq \delta \leqq 1/2$. *Let* $f(r)$ *be non-negative and continuous for* $0 < r \leqq 1/2$. *Furthermore, assume that* $f(r)$ *satisfies the integral equation*

$$\int_0^{1/2} K(\delta r, \omega) f(r)\, dr = l_R(\delta, \omega) \tag{15}$$

for all $\omega \geqq 0$. *Then* $f(r) \gg \ll r^{-1}$. ∎

For the proof of Lemma 4.3 see pp. 351—357 of [8]. Lemma 4.1 immediately follows from Lemma 4.3 and relations (12), (14), (15). ∎

The proof of Lemma 4.2 proceeds along the same lines as that of Lemma 4.1. We shall refer to [9], (see also [10]). Simple argument shows that for $r+s \leqq 1/2$

$$\int_{U^2} g(r, \mathbf{c};\ \mathbf{x})\, g(s, \mathbf{c};\ \mathbf{x})\, d\mathbf{c} = K(r, s, \omega(\mathbf{x}, \mathbf{y})), \tag{16}$$

where $K(r, s, \omega)$ denotes the area of the intersection of two discs of radii r, s with centres having distance ω. Similarly as above, by (16) we obtain

$$E(r, s) = \frac{1}{4} \sum_{\mathbf{p}' \in P} \sum_{\mathbf{p}'' \in P} K(r, s, \omega(\mathbf{p}', \mathbf{p}'')+ \tag{17}$$

$$+ \sum_{\mathbf{q}' \in Q} \sum_{\mathbf{q}'' \in Q} K(r, s, \omega(\mathbf{q}', \mathbf{q}'')) - \sum_{\mathbf{p} \in P} \sum_{\mathbf{q} \in Q} K(r, s, \omega(\mathbf{p}, \mathbf{q})).$$

Now we recall a result of [9].

Lemma 4.4. *Suppose that* $0 \leqq \delta \leqq 1/2$. *Suppose* $g(r)$ *is continuous for* $0 < r \leqq 1/2$ *and satisfies the integral equation*

$$\int_0^{1/2} K(\delta r, \omega) g(r)\, dr = \underset{r+s \leqq 1}{\int_0^1 \int_0^1} K(\delta r, \delta s, \omega) |r-s|^{-\alpha} \cdot |r+s|^{-(1-\alpha)}\, dr\, ds \tag{18}$$

for all $\omega \geqq 0$. *Then* $g(r) \gg \ll r^{-1-\alpha}$. ∎

The proof of Lemma 4.4 can be found on pp. 71—77 of [9]. Lemma 4.2 follows immediately from Lemma 4.4, and relations (14), (17), (18). ∎

5. Proof of Theorem 1.2, (ii)

Let $\xi_1, \xi_2, \ldots, \xi_N$ be independent random variables having values ± 1 each with probability 1/2. We define a "random two-colouring" as follows:

$$\zeta(i,j) = (-1)^i \xi_j, \quad 1 \leqq i, j \leqq N.$$

We claim that for this random two-colouring the event $E=\{$any tilted rectangle has "error"$\ll N^{1/2}(\log N)^{1/2}\}$ has probability $\geqq 1/2$. It is clear that this immediately yields Theorem 1.2, (ii).

Let B be any convex subset of the square $[1, N]^2$. Since B is convex, B intersects the j-th column $\{(i,j)\colon 1 \leqq i \leqq N\}$ of $[1, N]^2 \cap \mathbf{Z}^2$ in an interval. Therefore

$$\text{if} \quad |B \cap \{(i,j)\colon 1 \leqq i \leqq N\}| \quad \text{is even then} \quad \sum_{\substack{i=1\\(i,j)\in B}}^{N} \zeta(i,j) = 0, \quad \text{while}$$

$$\text{if} \quad |B \cap \{(i,j)\colon 1 \leqq i \leqq N\}| \quad \text{is odd then} \quad \sum_{\substack{i=1\\(i,j)\in B}}^{N} \zeta(i,j) = \pm \xi_j,$$

where the sign $\pm$ depends only on B. Hence the "error"

$$\sum_{\substack{i=1\\(i,j)\in B}}^{N} \sum_{j=1}^{N} \zeta(i,j)$$

of B equals $\sum_{j=1}^{N} c_j \xi_j$, where each constant c_j has one of the values $0, +1, -1$ depending only on B. Let $\sum_{j=1}^{N} |c_j| = n$. Using the well-known asymptotic properties of the binomial coefficients (see, e.g., [4]) we have

(19) $$\text{Prob}\,\{\text{the "error" of } B \text{ has absolute value} \geqq 2t\} =$$

$$= \text{Prob}\left\{\Big|\sum_{j=1}^{N} c_j \xi_j\Big| \geqq 2t\right\} = 2^{-n} \sum_{k\colon \left|k-\frac{n}{2}\right| \geqq t} \binom{n}{k} \leqq e^{-t^2/(2n)} \leqq e^{-t^2/(2N)}.$$

Let $\mathscr{B}$ denote the family of all convex subsets B of the square $[1, N]^2$ such that the boundary of B is a polygon with $\leqq 8$ sides. Given any tilted rectangle R, it is clear that the intersection $R \cap [1, N]^2$ belongs to $\mathscr{B}$. Though $\mathscr{B}$ is uncountable, it suffices to restrict ourselves to a subclass of $\mathscr{B}$ having cardinality $\ll N^{32}$. Indeed, given any $B \in \mathscr{B}$, one can easily find $B^* \in \mathscr{B}$ having the following properties: 1) each side of B^* contains at least two integer coordinate points 2) B and B^* contain exactly the same integer coordinate points of $[1, N]^2$. Let $\mathscr{B}^*$ denote the set of all elements of $\mathscr{B}$ wich have at least two integer coordinate points on each of their sides. A simple calculation shows that $|\mathscr{B}^*| \ll (N^2)^{16} = N^{32}$.

Now we are in the position to finish the proof. The above consideration and (19) yield

$$\text{Prob}\,\{\text{some tilted rectangle has „error" of absolute value } \geqq 2t\} =$$

$$= \text{Prob}\,\{\text{for some } B^*\in\mathcal{B}^*, \text{ the „error" of } B^* \text{ has absolute value } \geqq 2t\} \leqq$$

$$\leqq |\mathcal{B}^*|\cdot e^{-t^2/(2N)} \ll N^{32}e^{-t^2/(2N)}.$$

Choosing $t=c\cdot N^{1/2}\cdot(\log N)^{1/2}$ with a suitably large universal constant $c>0$, we get $N^{32}e^{-t^2/(2N)}\to 0$ as $N\to\infty$, and Theorem 1.2, (ii) follows. ▌

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